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AXISYMMETRIC CABLE-BAND MEMBRANES

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Abstract—A thin membrane formed by two sets of flexible members (bands and cables) is studied. The respective curvatures of the two sets face opposite directions which enables the membrane to be prestressed thereby acquiring necessary stiffness. The system belongs to the generic class of geometrically quasi-unstable systems whose characteristic feature is the interdependence between the geometry and the pattern of initial forces. This interdependence is investigated to establish the feasible configurations of the system, its basic properties and regularities.

INTRODUCTION

A large suspended cooling tower designed and built recently in West Germany[1] represents a new area for the application of cable systems. The skeleton of the structure is formed by three sets of cables: one set directed along the meridians and the remaining two inclined to a meridian at equal angles but in opposite directions. The meridional cables, on the one hand, and the two inclined sets of cables, on the other, have their respective curvatures facing opposite directions. Therefore, in prestressing, the two groups of cables develop mutual normal pressure thus considerably increasing the lateral stiffness of the system.

There exists, however, the possibility of increasing the lateral stiffness while having only two sets of structural members[2]. To this end, the two sets must be inclined to a meridian at different angles. As a result, the system remains axisymmetric, but loses its symmetry with respect to a meridional plane. The normal curvatures of the two sets become different and opposite which enables them to exert mutual normal pressure when being prestressed. One of the two sets can be made of finitely wide, overlapping, flexible bands so as to form a continuous axisymmetric cable-band membrane (Fig. 1). This, like a cable net, is a general-chain multi-freedom mechanical system.

Even when consisting of undeformable links, such systems normally allow a variety of configurations. However, under some exceptional combinations of the geometric parameters, a multifreedom system completely loses mobility and acquires a unique configuration called

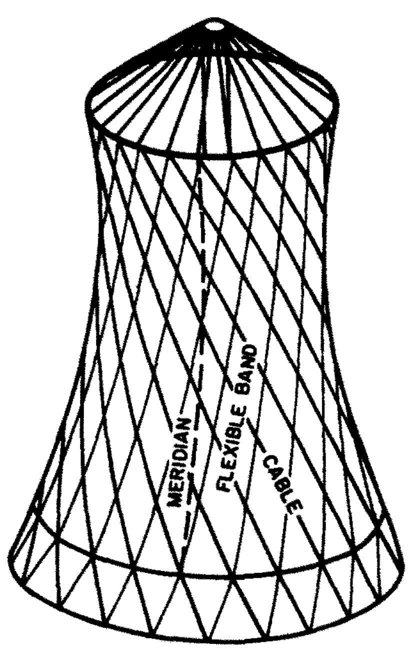

Fig. 1. Axisymmetric cable-band membrane.

static. The system is then classified as geometrically quasi-unstable[3]; it cannot be considered a geometrically stable structure since some finite loads would give rise to infinite internal forces i.e. to large displacements for a system made of a real material. The system can be stiffened by means of prestressing which is possible for (and only for) a static configuration.

Analysis of prestressed band-formed membranes under a general load[4] presumes a known state of prestress and, therefore, must be preceeded by structural synthesis. Its objective (and the subject of this study) is to confirm the feasibility of the above type of membrane and to determine the static configurations together with the associated initial force distributions. For these purposes, a cable-band system can be considered as a cable net. The latter was a subject for numerous investigations beginning with pioneering works[5]. However, these and most other works on the subject deal with a complete structural analysis and involve such aspects and assumptions that are unnecessary or unacceptable for the present study which is purely statical-geometric (i.e. does not involve constitutive relations). According to a general statical-geometrical theory of nets[6], a given net can be classified by examining its static vector, S_i, which is a known function of the geometric parameters of the net and is shown explicitly later. In order for a net to be static, it is necessary and sufficient that its static vector be gradient:

$$S_i = \frac{\partial S}{\partial x_i} \quad i = 1, 2. \tag{1}$$

This means that the two components, S_1 and S_2, of the static vector must be the partial derivatives of some scalar function, S, called the static potential. If this takes place, the initial forces in the two sets of cables are

$$T^*_\alpha = C\sigma_\beta \exp S \quad T^*_\beta = -C\sigma_\alpha \exp S. \tag{2}$$

Here C is an arbitrary constant determining the magnitude of the initial forces, σ_α and σ_β are the normal curvatures of the two respective sets of cables, T^*_α and T^*_β are forces referred to unit width strips $\mathrm{d}s_\beta = 1$ and $\mathrm{d}s_\alpha = 1$, respectively.

The meaning and the role of the static vector are as follows. Like any multifreedom system, a cable net is hyperstatic: three equilibrium equations of an infinitesimally small element of the net contain only two unknown forces. Requirement (1) imposed on the static vector is nothing but the condition of integrability of this overdetermined system of equations. The condition is instrumental in both analysis and synthesis of cable nets. First, it provides a straightforward way of determining by simple calculation whether a given net is static. In the latter case, the initial forces given by expressions (2) are the by-product of this calculation. In this way many important nets were identified as static, e.g. the net of principal curvatures of a second degree surface, the base net of an arbitrary translation surface, a conjugate geodesic net (Voss net), any orthogonal net on a minimal surface, and others.

Second, and more important, is the synthesizing feature of the concept of the static vector. Imposing some special requirements on the net sought but not identifying it completely permits expressions (1) to be used as equations from which all the statical-geometric attributes of the net of interest are determined. The present study examplifies the second feature: the static net sought is required to be axisymmetric, geodesic and torque-free while the problem statement calls for determining explicitly the entire set of such nets along with arbitrary elements (parameters or functions) governing the net configuration.

CHARACTERISTIC SIGN OF A STATIC AXISYMMETRIC GEODESIC NET

In a cable net with intersections not fixed, each cable acquires the form of a geodesic line on the net surface, so that the cables together form a geodesic net. Consider a geodesic net on a surface of revolution referred to its meridians and parallel circles. The direction unit vectors of the net lines, u and v, form angles α and β with the direction vector of a meridian (Fig. 2). The angles are measured from the meridian (positive if clockwise), depend only on the axial coordinate z and are the same for all cables of the respective set, i.e. the net possesses axial symmetry. For such an axisymmetric geodesic net, the static vector in the above reference

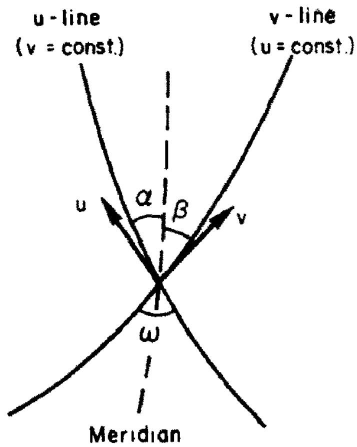

Fig. 2. Direction unit vectors of the net.

system takes the form[6]

$$S_i = -\frac{1}{\sin\omega}\left(\frac{\sigma_\alpha'}{\sigma_\alpha} v^1 u_i - \frac{\sigma_\beta'}{\sigma_\beta} u^1 v_i\right) + \frac{1}{\sin^2\omega}\left[\lambda_\alpha(u_i - v_i\cos\omega) - \lambda_\beta(v_i - v_i\cos\omega)\right], \tag{3}$$

where $\omega = \beta - \alpha$ is the net angle, the prime denotes a derivative with respect to z, λ_α and λ_β are the Tchebyshev curvatures of the net:

$$\lambda_\alpha = \frac{\beta'}{A}\cos\alpha + \frac{B'}{AB}\sin\alpha, \quad \lambda_\beta = \frac{\alpha'}{A}\cos\beta + \frac{B'}{AB}\sin\beta, \tag{4}$$

and u_i, u^i, v_i and v^i are the direction unit vectors of the net lines:

$$u_1 = -A\sin\alpha, \quad u^1 = \frac{\cos\alpha}{A}, \quad u_2 = B\cos\alpha, \quad u^2 = \frac{\sin\alpha}{B}, \tag{5}$$

v_i and v^i being obtainable by replacing α with β.

The Lamé parameters A and B for the coordinate lines (z, ϕ) of principal curvature can be taken, e.g. as

$$A = \sqrt{(1 + r')}, \quad B = r, \tag{6}$$

where $r = r(z)$ is the radius of revolution and ϕ is the polar angle.

Because of the axial symmetry, both the components S_1 and S_2 are functions of z only. If the static potential S exists, it also must be solely a function of z. Therefore, its derivative $\partial S/\partial\phi = S_2$ must be zero. Satisfying this condition separates the nets of interest from all axisymmetric geodesic nets. Prior to developing the equation $S_2 = 0$ in an explicit form, note that by virtue of the Clairaut theorem[7]

$$r\sin\alpha = b, \quad r\sin\beta = c, \tag{7}$$

where constants b and c (for bands and cables) define the two respective sets of geodesic lines on a surface of revolution. It is assumed that $c > |b|$ and, consequently, $\beta > |\alpha|$ everywhere in the net. This means that the bands are inclined at a smaller angle to a meridian than the cables, which makes sense for two practical reasons. First, in this case, the bands will be shorter than the cables. Second, the geodesic torsion of the band trajectories will be smaller than that of the cable trajectories which simplifies prestressing and results in a more favorable stress state of the bands (smaller torsion). Ultimately, the torsion induced in the bands by prestressing will approach zero if $\alpha \to 0$, because meridians are planar geodesic lines with zero geodesic torsion.

Differentiation of eqns (7) yields

$$\alpha' = -\frac{r'}{r}\tan\alpha \quad \beta' = -\frac{r'}{r}\tan\beta. \tag{8}$$

Setting $i = 2$ in eqn (3) and using expressions (4)–(8) results in

$$S_2 = -\frac{1}{A\sin\omega}\left[\left(\frac{\sigma_\alpha'}{\sigma_\alpha} - \frac{\sigma_\beta'}{\sigma_\beta}\right) r\cos\alpha\cos\beta - \left(\beta'\cos\alpha + \frac{r'}{r}\sin\alpha\right)c + \left(\alpha'\cos\beta + \frac{r'}{r}\sin\beta\right)b\right] = 0. \tag{9}$$

Integrating this equation leads to a closed-form relation characterizing the nets sought:

$$\frac{\sigma_\alpha}{\sigma_\beta} = C_1\frac{\cos\alpha}{\cos\beta}, \tag{10}$$

where C_1 is an arbitrary constant.

Of all axisymmetric geodesic nets, those and only those satisfying relation (10) are static.

INITIAL FORCES AND STATICAL-GEOMETRIC INTERRELATION

Upon specifying $i = 1$ in eqn (3) and making appropriate substitutions, the first component of the static vector acquires the form

$$S_1 = \frac{1}{\sin\omega}[(\ln\sigma_\alpha)'\tan\alpha - (\ln\sigma_\beta)'\tan\beta + 2(\beta - \alpha)'\cos\alpha\cos\beta]. \tag{11}$$

The curvature σ_α can be eliminated here by means of relation (10) and after some rearrangements the following expression of the static potential of the net is obtained by integration

$$S = \ln\frac{\sin\omega}{r\sigma_\beta\cos\alpha}. \tag{12}$$

According to eqns (2) and (10), the initial forces in the net are

$$T_b^* = C\frac{\sin\omega}{r\cos\alpha}, \quad T_c^* = -CC_1\frac{\sin\omega}{r\cos\beta}, \tag{13}$$

where subscripts b and c, denoting bands and cables, are used, when appropriate, in place of α and β, respectively.

As was previously mentioned, these forces relate to unit increments of respective complementary linear elements ds_c and ds_b. More physically apparent are forces, T_b and T_c, per unit increments, $dv = 1$ and $du = 1$, of the coordinates of net lines. The width of these strips varies but each contains a certain constant number of bands or cables. To obtain T_b and T_c, forces T_b^* and T_c^* must be multiplied by Lamé parameters B_n and A_n of the first quadratic form of the net. Introducing the coordinate differentials as

$$du = \frac{A}{B}\tan\beta\,dz - d\phi, \quad dv = \frac{A}{B}\tan\alpha\,dz - d\phi, \tag{14}$$

yields the following expressions for the Lamé parameters

$$A_n = \frac{r\cos\beta}{\sin\omega}, \quad B_n = \frac{r\cos\alpha}{\sin\omega}. \tag{15}$$

Forces T_b and T_c can now be evaluated using eqns (13) and (15):

$$T_b = B_n T_b^* = \frac{r\cos\alpha}{\sin\omega}\,\frac{C\sin\omega}{r\cos\alpha} = C, \quad T_c = A_n T_c^* = -C_1 C. \tag{16}$$

As it must be, both the forces are constant, i.e. do not vary along the bands and the cables. The reason is that the two sets interact only by pressing against each other in the direction of the normal to the surface. The intensity of this contact pressure is

$$P = \frac{T_b^*\sigma_\alpha}{\sin\omega} = -\frac{T_c^*\sigma_\beta}{\sin\omega} = \frac{C\sigma_\alpha}{r\cos\alpha} = C_1\frac{C\sigma_\beta}{r\cos\beta}. \tag{17}$$

By comparing eqns (10) and (16), the following relation can be established

$$\frac{\sigma_\alpha\cos\beta}{\sigma_\beta\cos\alpha} = -\frac{T_c}{T_b} = C_1. \tag{18}$$

This relation represents a pivotal link in the statical–geometric interrelation and is characteristic of the entire class of static axisymmetric geodesic nets.

To establish the role of the constant C_1 consider an element of a parallel circle with two adjacent tension members (Fig. 3). The circumferential shear component of the initial forces can be evaluated as

$$T_{12}^*\,ds_2 = T_b^*\,ds_c\sin\alpha + T_c^*\,ds_b\sin\beta. \tag{19}$$

As is readily seen from Fig. 3,

$$\frac{ds_b}{\cos\beta} = \frac{ds_c}{\cos\alpha} = \frac{ds_2}{\sin\omega}, \tag{20}$$

so that, in accordance with eqns (13) and (7),

$$T_{12}^* = \frac{C\sin\omega}{r}\left(\frac{\sin\alpha}{\sin\omega} - C_1\frac{\sin\beta}{\sin\omega}\right) = \frac{C}{r^2}(b - C_1 c). \tag{21}$$

The shear results in a torque moment

$$M_z = 2\pi r T_{12}^* r = 2\pi C(b - C_1 c). \tag{22}$$

which, of course, does not vary along the axial coordinate and equals the external (or reaction) moments acting in the opposite directions at the two end cross sections.

In a similar way, the meridional component of the initial forces can be determined:

$$T_1^*\,ds_2 = T_b^*\,ds_c\cos\alpha + T_c^*\,ds_b\cos\beta, \tag{23}$$

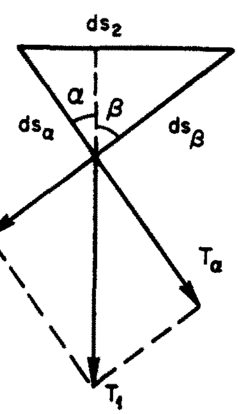

Fig. 3. Prestressing forces and their resultant.

wherefrom

$$T_1^* = \frac{C \sin \omega}{r} \left(\frac{\cos \alpha}{\sin \omega} - C_1 \frac{\cos \beta}{\sin \omega}\right) = \frac{C}{r}(\cos \alpha - C_1 \cos \beta). \tag{24}$$

PROPERTIES AND CONFIGURATION OF A TORQUE-FREE MEMBRANE

According to eqns (24) and (18), meridional force T_1^* cannot vanish if both T_b^* and T_c^* are positive. However, this is not the case with the shear force and, hence, with the torque moment. It would be highly advantageous if the membrane, as a free body, can be torque-free. This would mean that the circumferential shear components of the two initial forces cancel each other, so that the initial force resultant is meridional (Fig. 3). To this end, according to eqns (21) and (18), there must be

$$C_1 = \frac{b}{c} = \frac{\sin \alpha}{\sin \beta} = -\frac{T_c}{T_b}. \tag{25}$$

Since $\beta > 0$ and both the forces are assumed positive, it follows that for a torque-free membrane, $\alpha < 0$ and $b < 0$. This, most interesting, particular case will be considered now in more detail.

Necessary for further investigation is a well-known Euler formula[7] relating the normal curvature of a line on a surface to the principal curvatures, σ_1 and σ_2:

$$\sigma_\alpha = \sigma_1 \cos^2 \alpha + \sigma_2 \sin^2 \alpha. \tag{26}$$

Combining this equation with its counterpart for σ_β and taking into account eqns (10) and (25) yields

$$\frac{\sigma_1}{\sigma_2} = \tan \alpha \tan \beta. \tag{27}$$

The principal curvatures of a surface of revolution are given by

$$\sigma_1 = \frac{1}{R_1} = -\frac{r''}{(1 + r'^2)^{3/2}}, \quad \sigma_2 = \frac{1}{R_2} = \frac{1}{r(1 + r'^2)^{1/2}}. \tag{28}$$

Substituting these in eqn (27) and multiplying by r'/r yields

$$-\frac{r'r''}{1 + r'^2} = \frac{r'}{r} \tan \alpha \tan \beta. \tag{29}$$

Taking into account expressions (8), it can be checked by differentiation that the first integral of eqn (29) is

$$\frac{C_2}{\sqrt{(1 + r'^2)}} = r \sin(\alpha + \beta). \tag{30}$$

Introducing new variables, θ and x, allows this equation to be written as

$$C_2 \sin \theta = r \sin(\alpha + \beta) = x, \tag{31}$$

where θ is the angle between the normal to the surface and the axis of revolution (Fig. 4). At the equator, where $\theta = \pi/2$ and $r = a$, let $\alpha = \alpha_0$ and $\beta = \beta_0$. Then

$$C_2 = a \sin(\alpha_0 + \beta_0) = x_0. \tag{32}$$

Now some relevant force parameters of a torque-free membrane can be evaluated. Upon substituting C_1 in eqn (24), the meridional force becomes

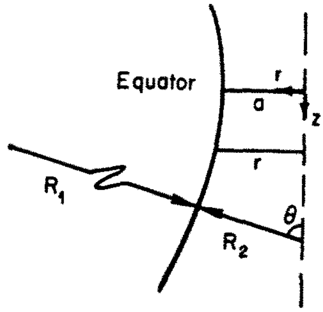

Fig. 4. Meridian of the net surface.

$$T_1^* = \frac{C \sin \omega}{r \sin \beta} = C_3 \sin \omega. \tag{33}$$

The axial and radial components of this force are, respectively

$$T_z^* = T_1^* \sin \theta = \frac{C_3}{C_2} \sin \omega r \sin (\alpha + \beta) = \frac{2C_3}{C_2} \frac{pq}{r} \tag{34}$$

$$T_r^* = T_1^* \cos \theta = \frac{C_3}{C_2} \sin \omega \sqrt{(C_2^2 - x^2)}, \tag{35}$$

where

$$p = c - b, \quad q = c + b. \tag{36}$$

The total axial force for the membrane equals

$$2\pi r T_z^* = \frac{4C_3\pi}{C_2} pq \tag{37}$$

and is the same in each cross section.

Equations governing the configuration of a torque-free membrane allow a closed-form solution. Using eqns (7) makes it possible to resolve eqn (31) with respect to r:

$$r = \frac{2bcx}{\sqrt{[(p^2 - x^2)(x^2 - q^2)]}}. \tag{38}$$

Now the explicit expression for the first radius of curvature can be obtained by employing eqn (31):

$$R_1 = \frac{1}{\cos \theta} \frac{dr}{d\theta} = \frac{2bcC_2(x^4 - p^2q^2)}{[(p^2 - x^2)(x^2 - q^2)]^{3/2}}. \tag{39}$$

Evaluation of the axial coordinate z is thereby reduced to quadrature:

$$z = \int_{\pi/2}^{\theta} R_1 \sin \theta \, d\theta = \frac{1}{x_0} \int_{x_0}^{x} \frac{R_1(x) x \, dx}{\sqrt{(x_0^2 - x^2)}}. \tag{40}$$

Equations (38) and (40) represent parametric equations of the meridian of the surface sought. Combined with eqns (10), (18) and (25) they provide exhaustive information on the statical-geometric interdependence for the system in consideration.

In applications, the following sequence of computations may be employed. Parameters b, c, and x_0 can be evaluated using eqns (7) and (31) as soon as radius r and angles α and β are assigned for a fixed θ. Then all the statical and geometric parameters of the system can be

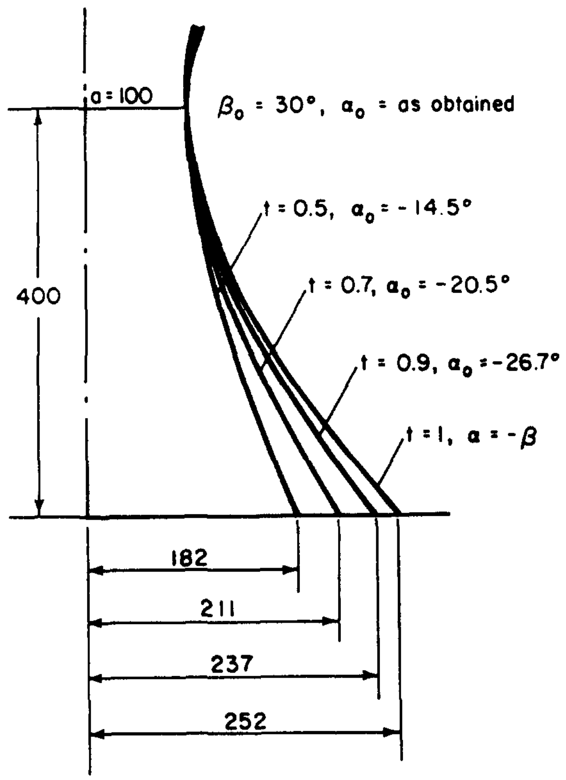

Fig. 5. Surface profiles for different prestressing force ratios.

determined as functions of z everywhere within the range of interest. A more practical alternative is to assign the ratio $t = T_c/T_b = -C_1$ instead of one of the angles (say, α). The most convenient location for the assignment of initial data is the equator because its plane is the plane of symmetry of the surface and the origin of the axial coordinate z. As a result, all the necessary computations can be made covering only the part of the range of interest with $z > 0$.

Thus, a torque-free axisymmetric geodesic net becomes uniquely determined by specifying the values $a = r(0)$, t and $\beta_0 = \beta(0)$. Obviously, the first one determines only the physical size of the net thus being simply a scale factor. The remaining two parameters govern the configuration of the net and provide a wide variety of forms from which to choose (Fig. 5).

Some basic regularities governing the system geometry were revealed in a computer-aided parametric study. In particular, it was found that, the larger the initial value β_0, the more rapidly the radius of revolution increases. Both the angles α and β vary considerably along the meridian. However, their relation is almost invariant and, by virtue of eqn (25), approaches the value of t for sufficiently small angles.

Shown in Fig. 5 are several profiles obtained for a fixed value of $a = 100$, one and the same angle $\beta_0 = 30°$ and different values of the ratio t. As the latter increases, the resulting profile widens and angles α and β become closer to each other in absolute value. This causes the normal curvatures of the members to decrease since the net lines are gradually approaching the asymptotic (zero normal curvature) lines of the surface. Accordingly, as can be seen from eqn (17), the normal contact pressure also diminishes until, at $t = 1$, the profile degenerates into a limiting hyperbola and the net lines become the linear generatrices of the one-sheet hyperboloid of revolution with zero contact pressure. From the standpoint of the lateral stiffness of the membrane, the optimum configuration is one with reasonably big normal curvatures of the members. For a fixed geometry, the contact pressure can be increased by inducing higher prestressing forces into the system.

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